Department of Mathematics and Statistics
Queen's University
2 0 2 4 - 2 0 2 5 Q U AL I F YI N G E XAM IN A TIO N S

The following is an excerpt from our Graduate Programs and Guidelines.

1.2.5 The Qualifying Exams

The Qualifying Exams are written exams covering core areas of mathematics and statistics. The student, in consultation with their supervisor(s), will select two core areas from a list of five, on which to be tested. The core areas are: Analysis; Algebra and Number Theory; Probability; Statistics; Geometry and Topology.

The exams will be offered twice per year, in May and January. Students beginning in September or May must take their exams in May of the following year while students beginning in January must take their exam by the following January.

The Objectives of the Qualifying Exams. The objectives of the exams are to assess both the candidate's knowledge of their broad research area and their competence in the core areas of mathematics and/or statistics related, possibly only peripherally, to their research.

The Exams. Each exam will consist of six questions and will last three hours. Questions will cover advanced undergraduate and first year graduate courses. Questions will be drawn from an exam bank[1] which will be available to doctoral students by the end of their first month of initial registration in the program

Evaluation of Exams. Exams will be evaluated by a specialist in the core area with the departmental Graduate Committee having final say on the result of each exam for each student. In order to earn full marks for a question the student must answer the entire question correctly. In order to pass an exam, students must receive full marks for four of the six questions or achieve a score of 5/6 through partial marks. Students who fail one or both exams will have one opportunity to rewrite the exam(s) at the next offering. Students must pass both selected exams to satisfy this requirement. No student shall be given more than two opportunities to pass any one qualifying exam.

Scheduling. Each exam will be offered only once during each exam period in May and January. Exams will be scheduled such that no student will be required to write both exams in one day. Exams will normally be held the third week of May and the third week of January when necessary. Exam dates and times will be announced at least fifteen working days in advance.

Responsibilities of Graduate Committee, student and supervisor with regard to the Exams.
It is the responsibility of the Graduate Committee to do the following:

1. review and update the exam bank by the end of August, annually;
2. provide core course instructors with the exam bank and remind them that there should be a correlation between the topics covered in a core course and the corresponding section of the exam bank;
3. remind new students and their supervisor(s) of the qualifying exams requirement and provide students with a link to the exam bank by the end of the student's first month of registration;
4. select the questions for each core exam;
5. set the exam dates and notify the students at least 15 days in advance of those dates;
6. arrange for the exams to be marked by a specialist in each core area, review the marked exams and inform the students of the outcome of each exam.

It is the responsibility of the student and supervisor to do the following:

1. decide upon the exams that the student will be required to take;
2. inform the Graduate Committee by the beginning of their 8th month in the program the exams the student will take.

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[1] Questions selected for an examination could have minor modifications made so as to avoid the possibility of a particular calculation being memorized.

ANALYSIS

Suggested textbooks:

Rajendra Bhatia, Fourier Analysis, Math. Assoc. America, 2005

Joseph Bak and Donald Newman, Complex Analysis, Springer 1997

Walter Rudin, Principles of Mathematical Analysis, McGraw-Hill, 1964

Walter Rudin, Real and Complex Analysis, McGraw-Hill, 1987

Queen's University, Department of Mathematics and Statistics

Qualifying Exam Problems: Analysis

1 Real Analysis

PROBLEM 1. Show that $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ is harmonic on $\mathbb{R}^3 \setminus (0, 0, 0)$.

PROBLEM 2. Find $\limsup_{x\to\infty} \big(\sin(x) + \sin(\pi x)\big)$. Justify your answer.

PROBLEM 3. Suppose $f \in C^1$ on $(0, 1)$ and for $0 < x < 1$ we have $|f'(x)| < 1$. Show that $\lim_{x\to 0^+} f(x)$ exits.

PROBLEM 4. Suppose $\sum_{n=1}^{\infty} a_n = s$. Show that $\sum_{n=1}^{\infty} a_n$ is Cesàro summable to s.

PROBLEM 5. Find $\displaystyle\lim_{a\to\infty} \int_{-a}^{a} \frac{\sin x}{x}\, dx$.

PROBLEM 6. Show that

$$\int_{-\infty}^{\infty} \frac{\tan t}{t}\, dt = \pi.$$

where the integral is interpreted as a Cauchy principal value.

PROBLEM 7. Suppose that $a_{m,n} \geq 0$. Show that

$$\sum_{m=1}^{\infty} \Big(\sum_{n=1}^{\infty} a_{m,n} \Big) = \sum_{n=1}^{\infty} \Big(\sum_{m=1}^{\infty} a_{m,n} \Big)$$

allowing for the possibility that one of or both of the sums is infinite.

PROBLEM 8. Let $[x]$ be the greatest integer less than or equal to x and $\{x\} = x - [x]$. So $\{1\} = 0$ and $\{\pi\} = 0.14159\ldots$. Find the limit $\lim_{n\to\infty} n\{n!e\}$.

PROBLEM 9. For $0 < a < b$ find the limit

$$\lim_{n\to\infty} \int_a^b \frac{\sin nx}{x}\, dx$$

when it exists.

PROBLEM 10. Find the limit

$$\lim_{t\to 0^+} t \int_{t^2}^{t} \frac{\cos x}{x^{3/2}}\, dx$$

when it exists. Justify your claims.

PROBLEM 11. Consider the rectangle R in the x-y plane with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b) where $a, b > 0$. Find the solid angle subtended by R at the point $c > 0$ on the z-axis. Evaluate any integrals you use.

PROBLEM 12. For $0 < a < b$ and $n > 0$ show that

$$\left| \int_a^b \frac{\cos x}{x^n}\, dx \right| \leq \frac{2}{a^n}$$

PROBLEM 13. Show that
$$\int_0^\infty \frac{e^{-x} - e^{-tx}}{x}\,dx = \log(t).$$

PROBLEM 14. Evaluate and compare the integrals
$$\int_0^1 \left[\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\right] dy \text{ and } \int_0^1 \left[\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\right] dx.$$

Should they be equal?

PROBLEM 15. Show that for $a, b, c > 0$ we have
$$4\,a\,b\,c \le a^4 + 2\,b^2 + c^4.$$

PROBLEM 16. Show that $\sqrt{n} \le \sqrt[n]{n!} \le \frac{n+1}{2}$ for $n \ge 1$.

PROBLEM 17. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $b_1 \ge b_2 \ge \cdots \ge b_n \ge 0$. Show that
$$mb_1 \le a_1b_1 + a_2b_2 + \cdots + a_nb_n \le Mb_1$$

where $m = \min\limits_{k=1,\ldots,n}\{a_1 + \cdots + a_k\}$ and $M = \max\limits_{k=1,\ldots,n}\{a_1 + \cdots + a_k\}$.

PROBLEM 18. For real numbers $s > 0$ define $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt$. Using this definition, show that for real numbers $s > 0$,
$$\Gamma(s+1) = s\Gamma(s).$$

PROBLEM 19. Let (X, d) be a metric space and $Y \subseteq X$ a closed non-empty subset. Let $\phi(x) = \inf\{d(y, x) \mid y \in Y\}$. Show that ϕ is continuous and $\varphi(x) = 0$ only for $x \in Y$.

PROBLEM 20. Let $a_0 > 0$ and $a_{n+1} = \frac{1}{1+a_n}$. Show that this sequence converges and find its limit.

PROBLEM 21. Let $f(a, b, c) = \int_0^1 (a + bx + cx^2)^2\,dx$ with $a, b, c \in \mathbb{R}$. Find the minimum value of f on $\{(a, b, c) \mid a + b + c = 1\}$.

PROBLEM 22. Evaluate $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+(x-y)^2+y^2)}\,dxdy$.

PROBLEM 23. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ converges for $x \in (0, \pi]$ and find its sum.

PROBLEM 24. Sketch the graph of $\sin(12345x) + \sin(12346x)$ and explain the shape of your graph.

PROBLEM 25. Compute the Fourier sine series for $f(\theta) = \cos\theta$ on the interval $[0, \pi]$.

PROBLEM 26. Let p be an integer greater than 1 and $0 \le x \le 1$ be a real number. Show that

(a) there is a sequence of integers $\{a_n\}_n$ such that $0 \le a_n < p$ and
$$x = \sum_{n\ge1} \frac{a_n}{p^n}$$

(b) this expansion is unique if and only if x is *not* of the form $\frac{q}{p^n}$ for some positive integers q and n,

(c) $x = \frac{q}{p^n}$ then x has exactly two expansions,

(d) what happens when $x = 1$?,

(e) every expansion $\sum_{n\geq 1} \frac{a_n}{p^n}$ converges to a real number $x \in [0,1]$.

PROBLEM 27. Suppose f is continuous on the interval $[a,b]$.

(a) Suppose $\epsilon > 0$ and for all $a < x < b$ we have $\limsup_{h\to 0^+} \dfrac{f(x+h)-f(x)}{h} \geq \epsilon$.

Show that for $a \leq x < y \leq b$ we have $f(x) \leq f(y)$.

(b) Suppose that for all $a < x < b$ we have $\limsup_{h\to 0^+} \dfrac{f(x+h)-f(x)}{h} > 0$.

Show that for $a \leq x < y \leq b$ we have $f(x) \leq f(y)$.

PROBLEM 28. Let A be an open subset of $\mathbb{R}$. For $x, y \in A$, say that $x \sim y$ if either $[x,y] \subset A$ or $[y,x] \subset A$. (Here, for $x \leq y$, $[x,y]$ denotes the closed interval with endpoints x and y.)

(a) Show that $\sim$ is an equivalence relation on A.

(b) Show that each equivalence class of this relation is an open interval.

(c) Show that A can be written as a countable disjoint union of open intervals.

PROBLEM 29. Let $\mathfrak{M}$ be a σ-algebra algebra of subsets of X and μ a measure on $\mathfrak{M}$. Recall that for a sequence of subsets $\{E_n\}_n$ we define $\limsup_n E_n$ to be

$$\bigcap_{n\geq 1} \bigcup_{k\geq n} E_k$$

and $\liminf_n E_n$ to be

$$\bigcup_{n\geq 1} \bigcap_{k\geq n} E_k.$$

Show that for $\{E_n\}_n \subset \mathfrak{M}$

(a) $\mu(\liminf_n E_n) \leq \liminf_n \mu(E_n)$, and

(b) $\limsup_n \mu(E_n) \leq \mu(\limsup_n E_n)$
provided $\mu(\cup_n E_n) < \infty$.

PROBLEM 30. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $\{E_n\}_n \subseteq \mathfrak{M}$ be a sequence of measurable sets such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Let $E = \{x \in X \mid x \in E_k \text{ for infinitely many } k\text{'s}\}$. Show that

(a) $E \in \mathfrak{M}$;

(b) $\mu(E) = 0$.

PROBLEM 31. Let $(\mathbb{R}, \mathfrak{M}, m)$ denote the measure space of the Lebesgue measurable subsets of the real line and m Lebesgue measure. By m^* we denote Lebesgue outer measure.

(a) Show that for any $A \subset \mathbb{R}$ there is $E \in \mathfrak{M}$ such that $A \subseteq E$ and $m^*(A) = m(E)$.

(b) Let $V \subset \mathbb{R}$ be in $\mathfrak{M}$ with $m(V) < \infty$. Show that for any subset $A \subseteq V$ there is $E \in \mathfrak{M}$ such that $E \subset A$ and $m^*(V \setminus A) = m(V \setminus E)$.

(c) Let $V \subset \mathbb{R}$ be in $\mathfrak{M}$ with $m(V) < \infty$. Let $E \subset V$ and suppose that $m(V) = m^*(E) + m^*(V \setminus E)$. Show that $E \in \mathfrak{M}$.

PROBLEM 32. Let $(X, \mathfrak{A}, \mu)$ be a measure space. Suppose that $1 \leq p < \infty$, that f, $g \in L^p(\mu)$ and that $\{f_n\}_n \subset L^p(\mu)$ is such that as $n \to \infty$, $\|f_n - f\|_p \to 0$ and $f_n(x) \to g(x)$ almost everywhere. Prove that $f(x) = g(x)$ almost everywhere.

PROBLEM 33. Let $(X, \mathfrak{A}, \mu)$ be a measure space and $f \in L^\infty(\mu)$. Let $R(f)$ be the set of all $z \in \mathbb{C}$ such that for all $\epsilon > 0$ we have

$$\mu(\{x \in X \mid |f(x) - z| < \epsilon\}) > 0.$$

(a) Show that $R(f)$ is compact. What is the relationship between $R(f)$ and $\|f\|_\infty$?

Let

$$A(f) = \Big\{\frac{1}{\mu(E)} \int_E f \, d\mu \mid E \in \mathfrak{A}, \mu(E) > 0\Big\}.$$

(b) Is $A(f)$ always a closed subset of $\mathbb{C}$? What are the relations between $A(f)$ and $R(f)$?

PROBLEM 34. Suppose φ is a measurable function from $\mathbb{R}$ to $\mathbb{R}$, that φ is integrable over finite intervals, and that φ is such that for every bounded real valued measurable function on $[0, 1]$ we have $\varphi(\int_0^1 f \, dm) \leq \int_0^1 \varphi \circ f \, dm$, where m denotes Lebesgue measure. Show that φ is convex.

PROBLEM 35. Let $(X, \mathfrak{A}, \mu)$ be a measure space with $\mu(X) < \infty$. Suppose $f \in L^\infty(\mu)$ and $\|f\|_\infty > 0$. Let $\alpha_n = \int |f|^n \, d\mu$. Show that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

PROBLEM 36. Recall that given a normed linear space $(X, \|\cdot\|)$, we let $X^* = \{\phi : X \to \mathbb{C} \mid \phi$ is bounded and linear$\}$. X^* is called the *dual* space of X.

Let $\{x_n\}_n \subseteq X$ be a sequence in X. We say that $\{x_n\}_n$ converges *strongly* to $x \in X$ if $\lim_n \|x_n - x\| = 0$. We say that $\{x_n\}_n$ converges *weakly* to $x \in X$ if for every $\phi \in X^*$ we have $\lim_n \phi(x_n) = \phi(x)$.

(a) Show that if $\{x_n\}_n$ converges weakly to x then $\|x\| \leq \limsup \|x_n\|$.

(b) Suppose that X is a Hilbert space. Show that if $\{x_n\}_n$ converges weakly to x and $\lim_n \|x_n\| = \|x\|$, then $\{x_n\}_n$ converges strongly to x.

PROBLEM 37. Let $(X, \|\cdot\|)$ be a finite dimensional normed linear space and $T : X \to \mathbb{C}$ a linear map. Show that there is C such that $|T(x)| \leq C\|x\|$ for all $x \in X$.

PROBLEM 38. For $m, n \geq 1$ let

$$a_{m,n} = \begin{cases} 2 - 2^{-m} & m = n \\ -2 + 2^{-m} & m = n + 1 \\ 0 & \text{otherwise} \end{cases}.$$

Compare $\sum_{m=1}^\infty \sum_{n=1}^\infty a_{m,n}$ with $\sum_{n=1}^\infty \sum_{m=1}^\infty a_{m,n}$. Are the hypotheses of Fubini's theorem satisfied?

PROBLEM 39. Let λ, μ, ν be measures on a measurable space $(X, \mathfrak{M})$.

(a) Suppose $\nu \ll \mu$ and $f : X \to [0, \infty]$ is measurable. Show that $\int f\, d\nu = \int f \frac{d\nu}{d\mu}\, d\mu$.

(b) Suppose $\nu \ll \mu$ and $\mu \ll \lambda$. Show that $\dfrac{d\nu}{d\lambda} = \dfrac{d\nu}{d\mu}\dfrac{d\mu}{d\lambda}$.

(c) Suppose $\nu \ll \mu$ and $\mu \ll \nu$. Show that $\dfrac{d\nu}{d\mu} = \left(\dfrac{d\mu}{d\nu}\right)^{-1}$.

PROBLEM 40. Let A and B be Lebesgue measurable sets on the real line, both of finite measure, and let χ_A and χ_B be their respective characteristic functions (i.e. $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise).

Show that the convolution $\chi_A * \chi_B$ defined by

$$\chi_A * \chi_B(x) = \int_{\mathbb{R}} \chi_A(y)\chi_B(x - y)\, dy$$

is a continuous function.

PROBLEM 41. Let $x \in (0, 1)$ have binary expansion

$$x = 0.a_1a_2a_2\cdots$$

You may use the fact that this expansion is unique for almost all x without proving it in this problem. For those x such that this expansion is unique define for $n = 1, 2, 3, ...$ the functions

$$r_n(x) = \begin{cases} 1 & \text{if } a_n = 1 \\ -1 & \text{if } a_n = 0. \end{cases}$$

For x such that its binary expansion is not unique, define $r_n(x) = 0$ for all n.

Prove if $f \in L^1(0, 1)$ then

$$\lim_{n\to\infty} \int_0^1 f(x)r_n(x)\, dx = 0.$$

2 Complex and Functional Analysis

PROBLEM 42. Let $G = \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Find a function holomorphic on G such that $f(0) = i$ and $1 + (f(z))^2 = z^2$.

PROBLEM 43. Suppose f and g are analytic function on $\mathbb{C}$ and for all z we have $|f(z)| \le |g(z)|$. What relation must there be between f and g?

PROBLEM 44. Let $L^2(0, 1)$ be the Hilbert space of square integrable functions on the interval $[0, 1]$ with Lebesgue measure. Define $K : L^2(0, 1) \to L^2(0, 1)$ by $K(f)(t) = \int_0^t (t - s)f(s)\, ds$. Show that $\|K\| < 1$. Given $g \in L^2(0, 1)$ find $f \in L^2(0, 1)$ such that $f = g + K(f)$.

PROBLEM 45. Let H be the space of rational functions with no poles on $\overline{\mathbb{D}} = \{z \mid |z| \le 1\}$. Give H the bilinear form $\langle f, g\rangle = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(z)\overline{g(z)}\frac{1}{z}\, dz$ where $\partial\mathbb{D}$ is the boundary of $\mathbb{D}$ and the integral is taken in the counter-clockwise direction. Show that this defines an inner product on H. Let T be the operator $\frac{d}{dz}$. Is T a bounded operator? Justify your answer.

PROBLEM 46. Let H be as in the previous question and $T(f)(z) = zf(z)$. Find the adjoint of T.

PROBLEM 47. Let $L^2(0,1)$ be the Hilbert space of square integrable functions on the interval $[0,1]$ with Lebesgue measure. Define $T : L^2(0,1) \to L^2(0,1)$ by $T(f)(t) = \int_0^t f(s)\,ds$. Show that $\|T\| < 1/\sqrt{2}$.

PROBLEM 48. Let $S = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$. Find an open set G containing S and f an analytic function on G such that $f(1/n) = (-1)^n/n$ for $n \in \mathbb{N}$, *or* prove that no such function exists.

PROBLEM 49. Let $\mathbb{D} = \{z \mid |z| < 1\}$ and suppose $f : \mathbb{D} \to \mathbb{C}$ is analytic, $f(0) = 0$, $f'(0) = 1$ and f is one-to-one. Show that there exists $g : \mathbb{D} \to \mathbb{C}$ which is analytic, one-to-one, and satisfies $(g(z))^2 = f(z^2)$ for $z \in \mathbb{D}$.

PROBLEM 50. Let H be a Hilbert space and $\{x_i\}_i \subset H$ a countable set of vectors that algebraically spans H, i.e. every vector in H is a linear combination of some finite subset of $\{x_i\}_i$. Show that H is finite dimensional.

PROBLEM 51. Let $P_r(\theta) = \dfrac{1-r^2}{1 - 2r\cos\theta + r^2}$ for $-\pi \le \theta \le \pi$ and $0 \le r < 1$. Show that $\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 2\pi$ for $0 \le r < 1$.

PROBLEM 52. Let $f(\theta) = |\theta|$ for $\theta \in [-\pi, \pi]$. Find the Fourier series of f. For which values of θ does the series converge to $f(\theta)$? Justify your answer.

PROBLEM 53. Let $f(z) = \dfrac{z-i}{z+i}$ for $z \in \mathbb{C}$. Find the image under f of $\{z \mid \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$.

PROBLEM 54. Find a fractional linear transformation f that maps $\{z \mid \mathrm{Im}(z) > 0\}$ onto $\{z \mid |z| < 1\}$ and such that $f(i) = 0$ and $\mathrm{Re}(f'(i)) = 0$.

PROBLEM 55. Let $\phi(t) = \dfrac{1}{\pi}\dfrac{1}{1+t^2}$ for $t \in \mathbb{R}$. Find the Fourier transform of ϕ. Justify your answer.

PROBLEM 56. (a) State the residue theorem.

(b) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{\frac{17}{4} + 2\cos\theta}$.

PROBLEM 57. (a) State the argument principle and use the residue theorem to prove it.

(b) Use Rouché's theorem to find how many roots of $z^5 + 2z^2 + 7z + 1$ lie inside $|z| = 2$? How many are real? Justify your answer.

PROBLEM 58. (a) Show that $f(z) = |z|^2$ is complex differentiable at $z = 0$ but nowhere else in $\mathbf{C}$.

(b) Let f be analytic on a punctured disc $0 < |z-a| < R$. Show that if the singularity $z = a$ is not removable then $e^{f(z)}$ has an essential singularity at $z = a$.

(c) Let f be analytic on $\mathbf{C}$ except for a finite number of poles. Show that if there exists k such that $|f(z)| < |z|^k$ for all sufficiently large z, then $f(z)$ is a rational function.

PROBLEM 59. (a) Show that $\sum_{n=0}^{\infty} \frac{z}{(1+z^2)^n}$ converges for all $z \in \mathbf{C}$ outside $|1+z^2| = 1$.

(b) Find analytic functions $f(z)$ defined on the unit disc that satisfy $f''(1/n) + f(1/n) = 0$ for $n = 2, 3, 4, \cdots$.

(c) State and prove the Fundamental Theorem of Algebra.

PROBLEM 60. (a) State the Cauchy integral formula.

(b) Evaluate the integrals $\int_\gamma \frac{1}{z(z-1)\cdots(z-k)}\,dz$ for $k = 0, 1, \cdots$ and $\int_\gamma \frac{(z-1)\cdots(z-k)}{z}\,dz$ for $k = 1, 2, \cdots$.

Here γ is a simple closed contour enclosing the points $0, 1, \cdots, k$ and oriented in the counter-clockwise direction.

(c) Find all linear fractional transformations that map $|z| < 1$ onto $|w| < 1$.

PROBLEM 61. Let $[0, z]$ denote the line in $\mathbb{C}$ starting at 0 and ending at z. Let $f(z) = \sqrt{\frac{2}{\pi}} \int_{[0,z]} \sin(w^2)\, dw$.

(a) Find the power series expansion of f at 0 and its radius of convergence.

(b) Let $g(z) = e^{1/z}$ for $z \neq 0$. What kind of singularity does g have at 0?

PROBLEM 62. For $z \in \mathbb{C}$ and $n \in \mathbb{Z}$, let $f_n(z)$ be the coefficient of w^n in the Laurent series expansion of $e^{z(w-1/w)/2}$. Show that $f_n(z) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - z\sin\theta)\, d\theta$.

PROBLEM 63. Let $a \in \mathbb{C}$ with $\mathrm{Re}(a) \neq 0$. Show that $\frac{2a}{\pi} \int_0^\infty (t^2 + a^2)^{-1}\, dt$ converges and find its value.

PROBABILITY

Suggested textbooks:

Probability and Measure, by Richard Dudley; Cambridge University Press, 2002

Probability Theory, by SRS Varadhan; Courant Institute, 2001

Probability and Measure, by R. Billingsley; John Wiley & Sons, 2008.

Probability Theory and Examples, by Rick Durrett; Cambridge University Press, 2010.

Random Processes for Engineers, by Bruce Hajek, Cambridge University Press, 2015

Queen's University, Department of Mathematics and Statistics

Qualifying Exam Problems: Probability

PROBLEM 1. Prove the following.

(a) Convergence in the almost sure sense implies convergence in probability.

(b) Convergence in the mean-square sense implies convergence in probability.

(c) If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

PROBLEM 2.

(a) Give an example where a sequence of random variables converges almost surely but not in mean-square.

(b) Give an example where a sequence of random variables converges in mean-square but not almost surely.

PROBLEM 3.

(a) Prove the Borel-Cantelli lemma, that is, the following result: Let $A_1, A_2, \dots, A_n, \dots \in \mathcal{F}$ be defined on a given probability space $(\Omega, \mathcal{F}, P)$

(i) If $\sum_n P(A_n)$ converges, then $P(\limsup_n A_n) = 0$.
(ii) If $\{A_n\}$ are independent and if $\sum P(A_n) = \infty$, then $P(\limsup_n A_n) = 1$.

(b) Consider a sequence of independent coin flips and let A_n, $n = 1, 2, \dots$, be the event that the nth coin flip is a head. What is the probability that there are infinitely many heads if $P(A_n) = 1/n^2$?

PROBLEM 4. Suppose that the sequence of random variables $(X_n)_{n\geq 1}$ converges to the random variable X in distribution as $n \to \infty$. Suppose that the sequence of random variables $(Y_n)_{n\geq 1}$ converges to the constant $c \neq 0$ in probability as $n \to \infty$. Prove that $(X_nY_n)_{n\geq 1}$ converges to cX in distribution as $n \to \infty$ and that $(X_n/Y_n)_{n\geq 1}$ converges to X/c in distribution as $n \to \infty$.

PROBLEM 5. Suppose that the sequence of random variables $(X_n)_{n\geq 1}$ converges to 0 in distribution as $n \to \infty$. Suppose that the sequence of random variables $(Y_n)_{n\geq 1}$ converges to the random variable Y in probability as $n \to \infty$. Consider a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $y \mapsto g(x, y)$ is a continuous function for all x and such that $x \mapsto g(x, y)$ is continuous at $x = 0$ for all y. Show that $(g(X_n, Y_n))_{n\geq 1}$ converges in probability to $g(0, Y)$

PROBLEM 6. A sequence of random variables $(X_n)_{n\geq 1}$ is said to be *completely convergent* to a random variable X if and only if

$$\sum_{n\geq 1} P(|X_n - X| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

Show that, for sequences of independent random variables, complete convergence is equivalent to convergence almost surely. Find a sequence of dependent random variables which converges almost surely but not completely.

PROBLEM 7. State the conclusions of the weak law of large numbers and the strong law of large numbers. Let $(X_n)_{n\geq 2}$ be a sequence of independent random variables such that

$$P(X_n = n) = P(X_n = -n) = \frac{1}{2n\log n}, \quad P(X_n = 0) = 1 - \frac{1}{n\log n}.$$

Show that this sequences satisfies the weak law of large numbers but not the strong law of large numbers.

PROBLEM 8.

(a) Let X and Y be real-valued random variables defined on a given probability space $(\Omega, \mathcal{F}, P)$. Show that $X+Y$ is also a random variable.

(b) Let $\mathcal{F}$ be a σ–field of subsets of a set Ω and let $A \in \mathcal{F}$. Prove that $\{A \cap B : B \in \mathcal{F}\}$ is a σ–field over A (that is a σ–field of subsets of A).

PROBLEM 9. Let $\mathcal{F}$ be a σ–field of subsets of a set Ω and suppose that $f_1, f_2, \ldots$ are real valued $\mathcal{F}$-measurable functions on Ω. Show that

(a) The functions $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$ are $\mathcal{F}$-measurable.

(b) If $\lim_n f_n$ exists everywhere, then $\lim_n f_n$ is $\mathcal{F}$-measurable.

(c) If f is $\mathcal{F}$-measurable, then the set $\{\omega : f_n(\omega) \to f(w)\}$ is in $\mathcal{F}$.

PROBLEM 10. Show that a function $F : \mathbb{R} \to \mathbb{R}$ is the cumulative distribution function of some random variable if and only if

(a) F is non-decreasing.

(b) $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$

(c) F is right-continuous; that is, if $x_n \downarrow x$, then $\lim_{x_n\to x} F(x_n) = F(x)$.

PROBLEM 11. Suppose X is a random variable having a continuous monotone increasing cumulative distribution function F. Find the distribution of the random variable $U = F(X)$.

PROBLEM 12.

(a) Is the set of rational numbers a Borel subset of $\mathbb{R}$? Explain your reasoning.

(b) Is every subset of $\mathbb{R}$ Borel? Is every subset of $\mathbb{R}$ Lebesgue measurable? Explain your reasoning.

PROBLEM 13. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a convex function, that is, $f(\alpha x+(1-\alpha)y) \le \alpha f(x)+(1-\alpha)f(y)$ for all $\alpha \in (0,1)$. If μ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $g : \mathbb{R} \to \mathbb{R}$ is a function such that g and $f(g)$ are μ-integrable, show that

$$f\Big(\int g(x)\mu(dx)\Big) \le \int f(g(x))\mu(dx)$$

This inequality is known as Jensen's Inequality.

PROBLEM 14. Let $\mathcal{I}$ be an arbitrary index set. A family of real random variables $\{X_\alpha, \alpha \in \mathcal{I}\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is said to be uniformly integrable if

$$\lim_{a\to\infty} \sup_{\alpha\in\mathcal{I}} E\big[|X_\alpha| 1_{\{|X_\alpha|\ge a\}}\big] = 0.$$

Show that $\{X_\alpha, \alpha \in \mathcal{I}\}$ is uniformly integrable if and only if $\sup_{\alpha\in\mathcal{I}} E\big[|X_\alpha|\big] < \infty$, and for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $A \in \mathcal{F}$ with $P(A) < \delta$, we have

$$\int_A |X_\alpha| dP < \epsilon \text{ for all } \alpha \in \mathcal{I}.$$

PROBLEM 15. Let $(X_i)_{i\ge1}$ be identically and independently distributed zero mean random variables with $0 < E(X_1^2) < \infty$ and let $S_n = \sum_{i=1}^n X_i$. Show that

$$P\left(\liminf_n \frac{1}{\sqrt{n}}|S_n| = 0\right) = 1 = P\left(\limsup_n \frac{1}{\sqrt{n}}|S_n| = \infty\right).$$

PROBLEM 16. Let $Y_1, Y_2, \ldots$ be a sequence of independent Bernoulli$(1/2)$ random variables and define $X_n = 2^n Y_1 Y_2 \cdots Y_n$ (thus X_n is the product of the Y_i multiplied by 2^n).

(a) Determine if X_n converges *in probability*, and if yes, to what limit.

(b) Does X_n converge *almost surely*, and if yes, to what limit?

PROBLEM 17. Let $Y_1, Y_2, \ldots$ be independent and identically distributed discrete random variables taking values in $\{0, 1, \ldots, 9\}$ with probabilities $P(Y_i = k) = 1/10$ for $k = 0, 1, \ldots, 9$. Let X be a continuous random variable that is uniformly distributed on the interval $[0, 1]$. Use either moment generating functions or characteristic functions to show that $X_n = \sum_{i=1}^{n} 10^{-i} Y_i$ converges to X in *distribution*.

PROBLEM 18. Let X and Y be two independent standard normal random variables. Determine the distribution of the random variable $U = X/Y$.

PROBLEM 19. The following method can be used to generate standard normal random variables using standard uniformly distributed random variables. Let X and Y be two independent random variables, each having uniform distribution on the interval $[0, 1]$. Consider new variables,

$$U = \sqrt{-2\ln X}\cos(2\pi Y) \quad \text{and} \quad V = \sqrt{-2\ln X}\sin(2\pi Y).$$

Find the joint probability density of U and V. *Hint:* $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$.

PROBLEM 20. Let $U_1, \ldots, U_n$ be independent random variables uniformly distributed on the interval $[-1, 1]$.

(a) Determine the limits in probability, as $n \to \infty$, of the following random variables:

$$\frac{U_1 + \cdots + U_n}{n}, \qquad \frac{U_1^2 + \cdots + U_n^2}{n}, \qquad \frac{U_1 + \cdots + U_n}{U_1^2 + \cdots + U_n^2}.$$

(b) What are the limits in distribution, as $n \to \infty$, of the following random variables:

$$\frac{U_1 + \cdots + U_n}{\sqrt{n}}, \qquad \frac{\sqrt{n}\,(U_1 + \cdots + U_n)}{U_1^2 + \cdots + U_n^2}, \qquad \frac{U_1 + \cdots + U_n}{\sqrt{U_1^2 + \cdots + U_n^2}}.$$

(c) What is the limit in distribution of the random variable $\dfrac{(U_1 + \cdots + U_n)^2}{U_1^2 + \cdots + U_n^2}$?

PROBLEM 21. Let X and Y be independent exponential random variables with parameters λ and μ, respectively. Show that $Z = \min\{X, Y\}$ is independent of the event $\{X < Y\}$. Find:

(a) $P(X = Z)$,

(b) the distributions of $U = \max\{X - Y, 0\}$, denoted as $(X - Y)^+$, and of $V = \max\{X, Y\} - \min\{X, Y\}$,

(c) $P(X \le t \le X + Y)$ for $t > 0$.

PROBLEM 22. Let X and Y be independent exponential random variables with parameters λ and μ respectively. Find the joint distribution of $S = X + Y$ and $R = X/(X + Y)$. What is the density of R?

PROBLEM 23. Let X and Y be independent Cauchy random variables. Find the density function of $Z = \alpha X + \beta Y$ assuming that $\alpha\beta \neq 0$.

PROBLEM 24. Let X be a uniform random variable on the interval $[0, 1]$ and let $0 < q < 1$. Show that the random variable $Y = 1 + \lfloor \log X / \log q \rfloor$ has a geometric distribution. Here $\lfloor t \rfloor = \max\{n \in \mathbb{Z} : n \le t\}$ denotes the floor function.

PROBLEM 25. Two points P and Q are picked independently and uniformly at random inside a ball $B \subset \mathbb{R}^3$. Consider the sphere S having centre P and radius $\|P - Q\|$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^3$. Compute the probability that S lies inside B.

PROBLEM 26. Let X and Y two random variables taking real values.

(a) Show that $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$E|XY| \le (E|X^p|)^{1/p}(E|Y^q|)^{1/q}.$$

(b) Show that if $p \ge 1$, then

$$(E(|X + Y|^p))^{1/p} \le (E|X^p|)^{1/p} + (E|Y^p|)^{1/p}.$$

PROBLEM 27. Let Z be a real-valued random variable. Show that the function $[0, \infty) \ni p \mapsto \log E|Z^p|$ is a convex function of p in every interval of values of p such that $E|Z^p| < \infty$. Deduce that

$$(E|Z^r|)^{1/r} \ge (E|Z^s|)^{1/s}$$

whenever $r \ge s > 0$.

PROBLEM 28. Let $(X_n)_{n \ge 1}$ be a sequence of real-valued random variables such that $X_n \le Y$ almost surely, where Y is a random variable with $E|Y| < \infty$. Show that

$$E\left(\limsup_{n\to\infty} X_n\right) \ge \limsup_{n\to\infty} E(X_n).$$

PROBLEM 29. Let 1_A denote the indicator function of the set A. Let X be a real-valued random variable. Prove that $E|X| < \infty$ if and only if the following holds: for all $\varepsilon > 0$, there exists $\delta > 0$, such that $E(1_A|X|) < \varepsilon$ for all measurable $A \subset \mathbb{R}$ with $P(A) < \delta$.

PROBLEM 30. Let $M_X(t)$ be the moment generating function of the random variable X. Define the *cumulant generating function $K_X(t)$ of the random variable X* as $K_X(t) = \log M_X(t)$. If M_X is finite in a neighbourhood of the origin, then K_X has a convergent Taylor series

$$K_X(t) = \sum_{n=1}^{\infty} \frac{k_n(X)}{n!} t^n,$$

where $k_n(X)$ is called the *nth cumulant of X*.

(a) Express $k_1(X)$, $k_2(X)$ and $k_3(X)$ in terms of the moments of X.

(b) Show that, if X and Y are independent, then $k_n(X + Y) = k_n(X) + k_n(Y)$ for every $n \ge 1$.

PROBLEM 31. Let X be a real-valued random variable. Define the moment generating function $M_X(t)$ of X and prove that

$$P(X \ge x) \le \inf_{t \ge 0} \left\{ e^{-tx} M_X(t) \right\}.$$

PROBLEM 32. For $n \ge 1$ consider the function

$$F_n(x) = x - \frac{\sin(2n\pi x)}{2n\pi}, \quad 0 \le x \le 1$$

(a) Show that F_n is the distribution function of some random variable, say X_n.

(b) Show that X_n has a density function, say f_n.

(c) Show that, as $n \to \infty$, F_n converges pointwise to the distribution function of a uniform random variable on $[0, 1]$, but that the density f_n does not converge to the density of a uniform random variable on $[0, 1]$.

PROBLEM 33. Let $(X_k)_{k \geq 1}$ be a sequence of independent identically distributed random variables such that $P(X_k = 1) = P(X_k = -1) = \frac{1}{2}$ for all $k \geq 1$. Prove that

$$\sqrt{\frac{3}{n^3}} \sum_{k=1}^{n} k X_k$$

converges in distribution, as $n \to \infty$, to a standard normal random variable $\mathcal{N}(0, 1)$.

PROBLEM 34. Give the definition of the characteristic function ϕ_X of a real-valued random variable X.

(a) Find the characteristic function of X^2, where X has distribution $\mathcal{N}(\mu, \sigma^2)$.

(b) Find the characteristic function of a Cauchy random variable Y.

(c) Which of the following functions are the characteristic function for some random variable? $\phi(t) = (1 + t^4)^{-1}$, $\phi(t) = e^{-t^4}$, $\phi(t) = \cos(t)$, $\phi(t) = \cos(t^2)$.

PROBLEM 35. The distribution of a random variable X is called *infinitely divisible* if, for all positive integers n, there exist n independent identically distributed random variables $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ such that X and $Y_1^{(n)} + \ldots + Y_n^{(n)}$ have the same distribution.

(a) Show that a Poisson distribution is infinitely divisible.

(b) Show that a gamma distribution is infinitely divisible.

(c) Let $\phi_X(t)$ be the characteristic function of an infinitely divisible real-valued random variable. Show that $\phi_X(t) \neq 0$ for all real t.

PROBLEM 36. For two distribution functions F and G define

$$d(F, G) := \inf\{\delta > 0 \ : \ F(x - \delta) - \delta \leq G(x) \leq F(x + \delta) + \delta \text{ for all } x \in \mathbb{R}\}.$$

Prove that d is a metric on the space of distribution functions.

PROBLEM 37. The *total variation distance* $d_{\mathrm{TV}}(X, Y)$ between two random variables X and Y is defined by

$$d_{\mathrm{TV}}(X, Y) := \sup_{\|u\|_\infty = 1} |E(u(X)) - E(u(Y))|,$$

where the supremum is over all measurable functions $u : \mathbb{R} \to \mathbb{R}$ such that $\|u\|_\infty = 1$ with, as usual, $\|u\|_\infty := \sup_{x \in \mathbb{R}} |u(x)|$. Show that $\lim_{n\to\infty} d_{\mathrm{TV}}(X_n, X) = 0$ implies that $X_n \to X$ in distribution as $n \to \infty$, but that the converse is false.

PROBLEM 38. Let $(X_k)_{k \geq 1}$ be a sequence of independent random variables with uniform distribution on $[-1, 1]$. Compute

$$\lim_{n\to\infty} P\left(\left|\sum_{k=1}^{n} X_k^{-1}\right| > \frac{\pi n}{2}\right).$$

PROBLEM 39. Let $(X_k)_{k\geq 1}$ be independent identically distributed integer-valued random variables. Let $S_n = \sum_{r=1}^n X_r$, with $S_0 = 0$, $Y_n = X_n + X_{n-1}$ with $X_0 = 0$, and $Z_n = \sum_{r=0}^n S_r$. Which of the following constitute Markov chains:

(a) the sequence $(S_n)_{n\geq 0}$,

(b) the sequence $(Y_n)_{n\geq 1}$,

(c) the sequence $(Z_n)_{n\geq 0}$,

(d) the sequence of pairs $((S_n, Z_n))_{n\geq 0}$.

PROBLEM 40. A particle performs a random walk on the vertices of a cube. At each step it remains where it is with probability $\frac{1}{4}$, or moves to one of its neighbouring vertices each having probability $\frac{1}{4}$. Let v and w be two diametrically opposite vertices. Assuming that the walk starts at v, find:

(a) the mean number of steps until its first return to v,

(b) the mean number of steps until its first visit to w,

(c) the mean number of visits to w before its first return to v.

PROBLEM 41. Consider a regular hexagon and join opposite corners by straight lines meeting at the point C. Consider the symmetric random walk performed by a particle on these 7 vertices, starting at vertex A. Find

(a) the probability of return to A without hitting C,

(b) the expected time to return to A,

(c) the expected number of visits to C before returning to A,

(d) the expected time to return to A given that there is no prior visit to C.

PROBLEM 42. Let a random walk on $\mathbb{Z}$ be described as follows. Let X_t denote the position (state) variable. Let the one-step transition probability be given by $P(X_{t+1} = b-1|X_t = b) = P(X_{t+1} = b+1|X_t = b) = 1/2$ for all $b \in \mathbb{Z}$ and all $t \in \mathbb{N}$.

Suppose the initial state is $a \in \mathbb{N}$. What is $P\left(\min(t > 0 : X_t = 0) < \infty \middle| X_0 = a\right)$?

PROBLEM 43. Consider a Markov chain $(X_t)_{t\geq 1}$ with a countable state space $\mathbb{X}$, defined on a probability space $(\Omega, \mathcal{F}, P)$. For any set $A \subset \mathbb{X}$, the occupation time η_A is the number of visits of $\{X_t, t \in \mathbb{Z}_+\}$ to set A given by

$$\eta_A = \sum_{t=1}^{\infty} 1_{\{X_t \in A\}},$$

where 1_E denotes the indicator function for an event E, that is, it takes the value 1 when E takes place, and is otherwise 0. Let $\tau_A = \min\{t > 0 : X_t \in A\}$, be the first (entry) time that the state visits the set A.

Prove that if $P(\tau_{\{i\}} < \infty | X_0 = i) = 1$ for some $i \in \mathbb{X}$, then $P(\eta_{\{i\}} = \infty | X_0 = i) = 1$.

PROBLEM 44. Let X be a random variable such that $|X| < K$ almost surely for some positive constant K. Let $Y_n = X + W_n$ for $n = 1, 2 \ldots$, where W_n is some "noise" process (not necessarily independent of X).

Let $\mathcal{F}_n$ be the σ-field generated by $Y_0, Y_1, \ldots, Y_n$.

Does $\lim_{n\to\infty} E[X|\mathcal{F}_n]$ exist almost surely? Prove your statement.

PROBLEM 45. Let X_n be a submartingale defined on a filtration $\mathcal{F}_n$ and suppose that $\sup_{n\geq 0} E[|X_n|] < \infty$. Show that $\lim_{n\to\infty} X_n = X$ exists almost surely.

PROBLEM 46. Let X, W be independent real Gaussian random variables with mean 0 and variance 1. Let $Y = X + W$. Suppose that we wish to find

$$\gamma^* = \inf_g E[|X - g(Y)|^2]$$

over all Borel measurable functions $g : \mathbb{R} \to \mathbb{R}$.

Find γ^* and an optimal function g.

PROBLEM 47. Let X, Y, Z be real random variables with finite second moments, defined on a common probability space. Is it true that

$$E[(X - E[X|Y, Z])^2] \leq E[(X - E[X|Y])^2]?$$

Either prove it or give a counterexample.